Modeling and Simulating Complex Materials subject to Frictional Contact

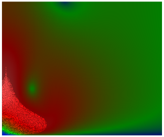
Application to fibrous and granular materials

Gilles Daviet

Advised by Florence Bertails-Decoubes

Équipe-projet Bipop

Inria — Laboratoire Jean Kuntzmann — Université Grenoble Alpes



December 15, 2016 — Montbonnot, France

Complex materials

- A large number of individual constituants
- Interact mostly through contacts with dry friction

Complex materials

- A large number of individual constituants
- Interact mostly through contacts with dry friction
- Emergence of complex behavior

Transition from solid to liquid: landslides

Kaikoura, New Zealand, November 14 2016 ©GNS Science, RNZ

Computer graphics for feature films

- Complex materials tedious to animate by hand
- Qualitative prediction rather than quantitative
- Requires robustness and computational efficiency
- Avoid artifacts such as creeping motion or jittering

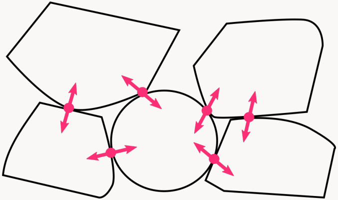
Importance of dry friction for visual apperance

Numerical simulation

How to simulate complex materials numerically?

Discrete Element Modeling

Simulate each constituant individually, and the interactions between them



- Controllabilty
- Computational cost

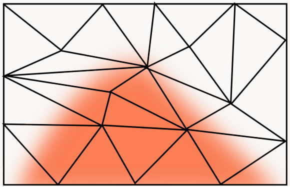
Numerical simulation

How to simulate complex materials numerically?

Continuum approach

Considers the “averaged” behavior of many constituants (zoom-out).

- E.g. Navier-Stokes for Newtonian fluids



- 🙂 Cost no longer depends on the system’s size
- Inhomogeneities must be relatively small
- Macroscopic model has to be derived

Outline

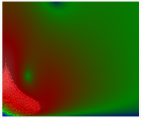
1. Efficient simulation of frictional contacts in DEM

- Application to hair simulation
- Presented at Siggraph Asia 2011

2. Continuum simulation of dry granular materials

- Dense case: JNNFM 2016
- General case: Siggraph 2016

3. Continuum simulation of granular materials in a Newtonian fluid



- Exploratory 2D work
- Submitted to “Powder and Grains 2017”

Outline

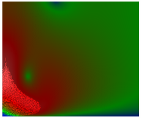
1. Efficient simulation of frictional contacts in DEM

- Application to hair simulation
- Presented at Siggraph Asia 2011

2. Continuum simulation of dry granular materials

- Dense case: JNNFM 2016
- General case: Siggraph 2016

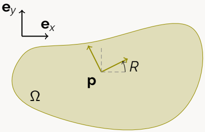
3. Continuum simulation of granular materials in a Newtonian fluid



- Exploratory 2D work
- Submitted to "Powder and Grains 2017"

Discrete Element Modeling with contacts

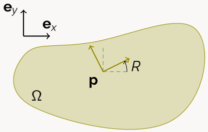
1. Choice of a mechanical model for each constituant



- Spatial discretization
- Internal and external forces
- Time integration

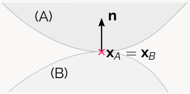
Discrete Element Modeling with contacts

1. Choice of a mechanical model for each constituant



- Spatial discretization
- Internal and external forces
- Time integration

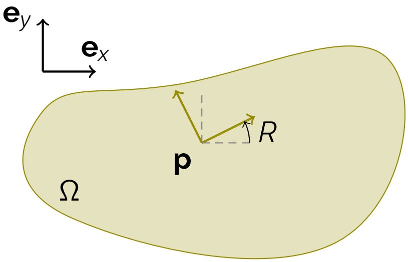
2. Choice of a mechanical model for the contacts



- Frictional contact law
- Numerical integration (with contacts)

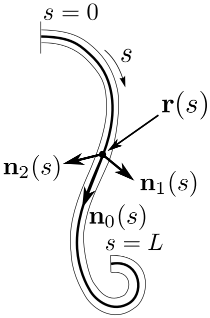
Discrete mechanical model

Example: rigid-body

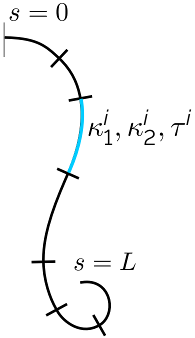


Discrete mechanical model

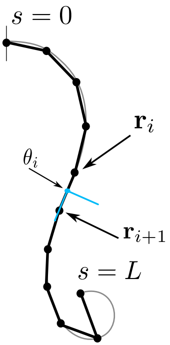
Example: slender inextensible elastic rod



Kirchhoff rod (continuous model)



Super-helix model [Bertails et al. 2006]



Discrete Elastic Rods model [Bergou et al. 2008]

Time integration

Initial value problem

Continuous-time equations

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{v}$$

$$M(\mathbf{q})\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = f(t, \mathbf{q}, \mathbf{v})$$

$$\mathbf{q}(t^0) = \mathbf{q}^0$$

$$\mathbf{v}(t^0) = \mathbf{v}^0$$

- $\mathbf{q}$ generalized coordinates
- $\mathbf{v}$ generalized velocities

Time integration

Initial value problem

Continuous-time equations

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} &= \mathbf{v} \\
M(\mathbf{q})\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= f(t, \mathbf{q}, \mathbf{v}) + \left(\frac{\partial C}{\partial \mathbf{q}}\right)^{\top} \lambda \\
\mathbf{q}(t^0) &= \mathbf{q}^0 \\
\mathbf{v}(t^0) &= \mathbf{v}^0 \\
C(\mathbf{q}, t) &= \mathbf{0}
\end{aligned}
$$

- $\mathbf{q}$ generalized coordinates
- $\mathbf{v}$ generalized velocities

Discrete-time integration

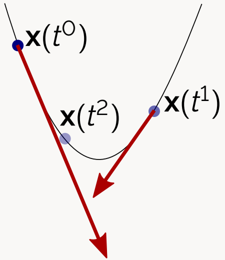
Compute $\mathbf{q}(t + \Delta_t), \mathbf{v}(t + \Delta_t)$ from $\mathbf{q}(t)$ and $\mathbf{v}(t)$

Discrete-time integration

Compute $\mathbf{q}(t + \Delta_t), \mathbf{v}(t + \Delta_t)$ from $\mathbf{q}(t)$ and $\mathbf{v}(t)$

Explicit Euler

- Evaluate forces using positions and velocities from beginning of time-step
- 😊 Straightforward to implement
- ☹ Prone to parasitic oscillations
  $\implies$ requires small timesteps

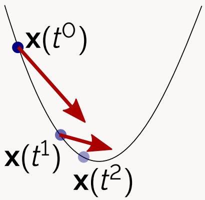


Discrete-time integration

Compute $\mathbf{q}(t + \Delta_t), \mathbf{v}(t + \Delta_t)$ from $\mathbf{q}(t)$ and $\mathbf{v}(t)$

Implicit Euler

- Predict forces at the end of the timestep $t + \Delta_t$
- 🙂 Stable
- 🙂 End-of-step position satifies kinematic constraints
- ☹ More expensive (root-finding algorithm)



Discrete-time integration

Compute $\mathbf{q}(t+\Delta_t), \mathbf{v}(t+\Delta_t)$ from $\mathbf{q}(t)$ and $\mathbf{v}(t)$
Using an iterative approach: solve (one or more) linear systems

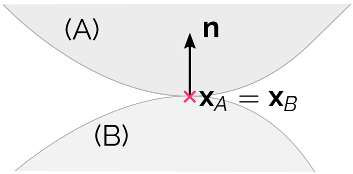
Without kinematic constraints

$$M\mathbf{v}(t^{k+1}) = \mathbf{f}$$

With kinematic constraints

$$\begin{cases} M\mathbf{v}(t^{k+1}) = \mathbf{f} + C^\top \lambda \\ C\mathbf{v}(t^{k+1}) = \mathbf{k} \end{cases}$$

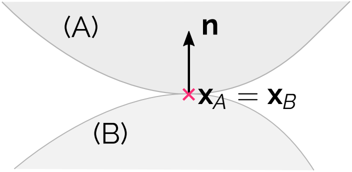
Contacts



Hypothesis

1. At most two objects, A et B
2. Smooth contact surface: well-defined normal $\mathbf{n}$

Contacts



Hypothesis

1. At most two objects, A et B
2. Smooth contact surface: well-defined normal $\mathbf{n}$

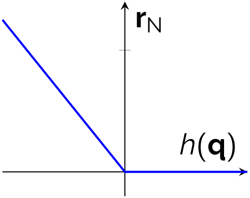
$\rightarrow$ local basis in which to express

- the gap function : $h(\mathbf{q}) = (\mathbf{x}_A - \mathbf{x}_B) \cdot \mathbf{n}$
  - Contact while $h(\mathbf{q}) \leq 0$
- the relative velocity $\mathbf{u}$ A/B
- the contact force $\mathbf{r}$ $B \rightarrow A$

Compliance

Heuristically derived from elastic response due to local deformation near contact point with force proportional to interpenetration distance

$$\mathbf{r}_N = \frac{1}{\xi} \max(0, -h(\mathbf{q}))$$

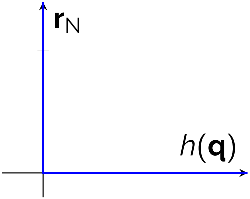


Drawbacks

- Non-zero penetration
- Leads to stiff equations hard to solve numerically
  - Explicit $\implies$ parasitic oscillations
  - Implicit $\implies$ ill-conditioned

Rigid contact assumption

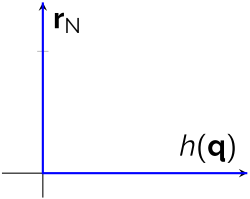
$$\begin{cases} h(\mathbf{q}) \geq 0 \\ h(\mathbf{q}) > 0 \implies \mathbf{r}_{\mathrm{N}} = 0 \\ h(\mathbf{q}) = 0 \implies \mathbf{r}_{\mathrm{N}} \geq 0 \end{cases}$$



- ☺ Penetration-free
- ☺ Does not introduce any new timescale

Rigid contact assumption

$$\begin{cases} h(\mathbf{q}) \geq 0 \\ h(\mathbf{q}) > 0 \implies \mathbf{r}_N = 0 \\ h(\mathbf{q}) = 0 \implies \mathbf{r}_N \geq 0 \end{cases}$$



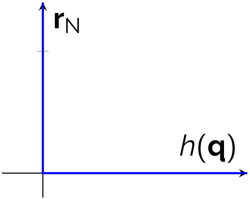
- Penetration-free
- Does not introduce any new timescale

Complementarity notation

$$0 \leq \mathbf{r}_N \perp h(\mathbf{q}) \geq 0$$

Rigid contact assumption

$$\begin{cases} h(\mathbf{q}) \geq 0 \\ h(\mathbf{q}) > 0 \implies \mathbf{r}_\mathrm{N} = 0 \\ h(\mathbf{q}) = 0 \implies \mathbf{r}_\mathrm{N} \geq 0 \end{cases}$$



- 🙂 Penetration-free
- 🙂 Does not introduce any new timescale

Complementarity notation

$$0 \leq \mathbf{r}_\mathrm{N} \perp h(\mathbf{q}) \geq 0$$

Assuming inelastic impacts: (no rebound)

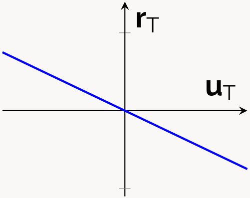
$$0 \leq \mathbf{r}_\mathrm{N} \perp \mathbf{u}_\mathrm{N} \geq 0$$

Friction

"Viscous" (fluid) friction

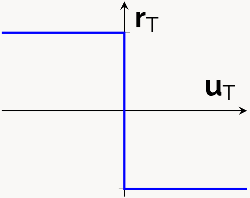
$$\mathbf{r}_\mathrm{T} = -\eta(|\mathbf{u}|)\mathbf{u}_\mathrm{T}$$

- Opposed to velocity
- Drops to zero when velocity does
- Never comes to rest



"Dry" (solid) friction

- Opposed to velocity
- May persist when velocity is zero
- Now sliding while below threshold

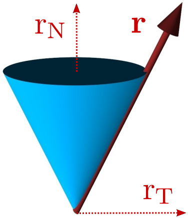


Coulomb frictional contact law (1780)

Dry friction with threshold proportional to applied load:
Contact force **r** in second-order cone K_μ,

$$K_\mu = \{\|\mathbf{r}_\mathrm{T}\| \leq \mu \mathbf{r}_\mathrm{N}\} \subset \mathbb{R}^3,$$

with μ the friction coefficient.



Coulomb frictional contact law (1780)

Dry friction with threshold proportional to applied load:
Contact force $\mathbf{r}$ in second-order cone K_μ,

$$K_\mu = \{\|\mathbf{r}_\mathsf{T}\| \leq \mu \mathbf{r}_\mathsf{N}\} \subset \mathbb{R}^3,$$

with μ the friction coefficient.

$$(\mathbf{u}, \mathbf{r}) \in C(\mu) \iff$$

$$\left\{ \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right.$$

Coulomb frictional contact law (1780)

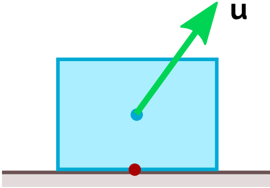
Dry friction with threshold proportional to applied load:
Contact force $\mathbf{r}$ in second-order cone K_μ,

$$K_\mu = \{\|\mathbf{r}_\mathsf{T}\| \leq \mu \mathbf{r}_\mathsf{N}\} \subset \mathbb{R}^3,$$

with μ the friction coefficient.

$$(\mathbf{u}, \mathbf{r}) \in C(\mu) \iff$$

$$\begin{cases} \text{take-off} \quad \mathbf{r} = 0 \text{ and } \mathbf{u}_\mathsf{N} > 0 \\ \\ \\ \end{cases}$$



Coulomb frictional contact law (1780)

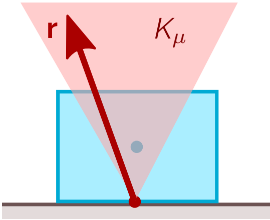
Dry friction with threshold proportional to applied load:
Contact force **r** in second-order cone K_μ,

$$K_\mu = \{\|\mathbf{r}_\mathrm{T}\| \leq \mu \mathbf{r}_\mathrm{N}\} \subset \mathbb{R}^3,$$

with μ the friction coefficient.

$$(\mathbf{u}, \mathbf{r}) \in C(\mu) \iff$$

$$\begin{cases} \text{take-off} & \mathbf{r} = 0 \text{ and } \mathbf{u}_\mathrm{N} > 0 \\ \text{sticking} & \mathbf{r} \in K_\mu \text{ and } \mathbf{u} = 0 \\ & \end{cases}$$



Coulomb frictional contact law (1780)

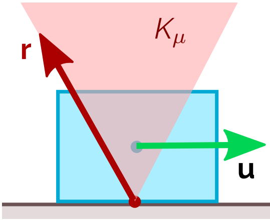
Dry friction with threshold proportional to applied load:
Contact force **r** in second-order cone K_μ,

$$K_\mu = \{\|\mathbf{r}_\mathsf{T}\| \leq \mu \mathbf{r}_\mathsf{N}\} \subset \mathbb{R}^3,$$

with μ the friction coefficient.

$$(\mathbf{u}, \mathbf{r}) \in C(\mu) \iff$$

$$\begin{cases} \text{take-off} & \mathbf{r} = 0 \text{ and } \mathbf{u}_\mathsf{N} > 0 \\ \text{sticking} & \mathbf{r} \in K_\mu \text{ and } \mathbf{u} = 0 \\ \text{sliding} & \mathbf{r} \in \partial K_\mu \setminus 0,\ \mathbf{u}_\mathsf{N} = 0 \\ & \text{and } \exists \alpha \geq 0,\ \mathbf{u}_\mathsf{T} = -\alpha\, \mathbf{r}_\mathsf{T} \end{cases}$$



Constraints inside timestepping scheme

Unconstrained dynamics

$$M\mathbf{v} = \mathbf{f}$$

Non-smooth contact dynamics (Moreau–Jean)

Discrete Coulomb Friction Problem (DCFP):

$$\begin{cases} M\mathbf{v} = \mathbf{f} + H^\top \mathbf{r} \\ \mathbf{u} = H\mathbf{v} + \mathbf{w} \\ (\mathbf{u}_i, \mathbf{r}_i) \in C(\mu_i) \quad \forall 1 \leq i \leq n \end{cases}$$

with $H := \frac{\partial \mathbf{u}}{\partial \mathbf{v}}$. $\mathbf{r}$ impulse (integrated force over timestep).

Solving the DCFP

Coulomb friction problem: Non-convex, possibly non-existence (if forcing term) or non-unicity of solutions.

- Disjunctive formulation not convenient (3^n cases to check)

Solving the DCFP

Coulomb friction problem: Non-convex, possibly non-existence (if forcing term) or non-unicity of solutions.

- Disjunctive formulation not convenient (3^n cases to check)
- Functional reformulations $f(\mathbf{r}) = 0$
  - f non-differentiable (e.g. Alart–Curnier)
  - Potentially quadratic convergence near solution
  - In practice: not very robust

Solving the DCFP

Coulomb friction problem: Non-convex, possibly non-existence (if forcing term) or non-unicity of solutions.

- Disjunctive formulation not convenient (3^n cases to check)
- Functional reformulations $f(\mathbf{r}) = 0$
  - f non-differentiable (e.g. Alart–Curnier)
  - Potentially quadratic convergence near solution
  - In practice: not very robust
- Optimization-based

$$(\mathbf{u}_i, \mathbf{r}_i) \in C(\mu_i) \iff K_{\frac{1}{\mu_i}} \ni \mathbf{u}_i + \mu_i \|\mathbf{u}_{i\mathrm{T}}\| \mathbf{n}_i \perp \mathbf{r}_i \in K_{\mu_i}$$

  - DCFP “close” to Second-Order Cone Quadatic Program
  - Outer fixed-point loop (Haslinger, Renouf, Cadoux) or descent direction modification
  - e.g. Projected Gradient, Gauss–Seidel

Gauss–Seidel strategy

Adaptation of block-coordinate descent to DCFP

- Solve contact-by-contact
- Slow asymptotic convergence
- ... but fast approximate solution $\implies$ good for graphics (and others)

Gauss–Seidel strategy

Adaptation of block-coordinate descent to DCFP

- Solve contact-by-contact
- Slow asymptotic convergence
- … but fast approximate solution $\implies$ good for graphics (and others)
- Requires one-contact solver

Local Gauss–Seidel solver

Local problem

$$\begin{cases} \mathbf{u}_i = W\mathbf{r}_i + \mathbf{b}_i \\ (\mathbf{u}_i, \mathbf{r}_i) \in C(\mu_i) \subset \mathbb{R}^d \times \mathbb{R}^d, \quad d = 2 \text{ or } 3 \end{cases}$$

Problem: For Super-Helix model matrix W may be ill-conditioned

$\implies$ Need robust local solver (otherwise GS diverges)

Standard local solvers based on functional formulation fail too often

Analytical local solver

For 1 contact: only three cases, disjunctive formulation becomes tractable

- "Take-off" and "sticking" case trivial to check
- "Sliding case": solutions in roots of degree-4 polynomial
  - Analytical solution (e.g. Ferrari algorithm)
  - Eigenvalues of companion matrix

Analytical local solver

For 1 contact: only three cases, disjunctive formulation becomes tractable

- "Take-off" and "sticking" case trivial to check
- "Sliding case": solutions in roots of degree-4 polynomial
  - Analytical solution (e.g. Ferrari algorithm)
  - Eigenvalues of companion matrix

- 🙂 If there exists a solution: we get it
- ☹ If local problem does not possesses a solution: we're stuck

Newton-based solver

Solution: use analytical in combination with Newton solver. We use Second-Order Cone Fischer–Burmeister function (Fukushima et al. 2001)

- “smoother” than projection-based ones (e.g. Alart-Curnier)
- Always yield an approximate solution

Performance comparisons

on 306 one-step problems

Note that we could not successfully run our full-scale simulations with any method other than our approach.

Local solver	Failure rate (%)	$\overline{\text{GS Iters}}$	$\overline{\text{Time}}$ (ms)
Newton FB	4.9	72	484
Enumerative	1	67	1044
Our method	**0**	**41**	**312**

- Hybrid approach improves both robustness and time efficiency

Full hair simulations

2000 super-helices — 4 mins / frame

Limitations

- Scalability
  - Contacts scale super-linearly with number of fibers
  - Contact solver cost scales super-linearly with number of contacts
  - Gauss–Seidel inherently sequential
  - $\implies$ Cannot simulate full groom

Limitations

- Scalability
  - Contacts scale super-linearly with number of fibers
  - Contact solver cost scales super-linearly with number of contacts
  - Gauss–Seidel inherently sequential
  - $\Longrightarrow$ Cannot simulate full groom
- Lots of phenomena not modeled yet
  - Friction anisotropy ?
  - Electrostatic forces ?
  - Interaction with air ?

Outline

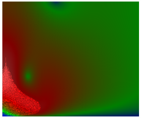
1. Efficient simulation of frictional contacts in DEM

- Application to hair simulation
- Presented at Siggraph Asia 2011

2. Continuum simulation of dry granular materials

- Dense case: JNNFM 2016
- General case: Siggraph 2016

3. Continuum simulation of granular materials in a Newtonian fluid



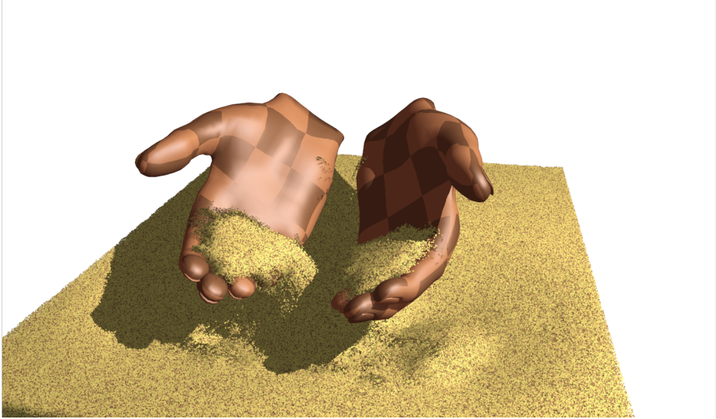
- Exploratory 2D work
- Submitted to “Powder and Grains 2017”

We want to simulate much larger systems.

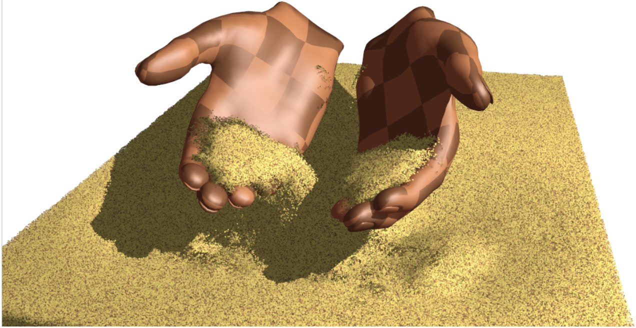
- We go back to simpler constituants: monodisperse spherical grains
- Macroscopic models exist for granulars (quantitative in certain scenarios [Jop 2006])
- Intuition: slope of sand heap does not depend of number of grains (a twice bigger heap will maintain the same slope)

Example simulation

20M rendered particles – 30s per frame



Granular regimes



Continuum mechanics

$\boldsymbol{u}$ velocity field, ρ density field
Conservation of momentum:

$$\underbrace{\rho\frac{D\boldsymbol{u}}{Dt}}_{\text{Inertial terms}} - \nabla \cdot \left[\underbrace{\boldsymbol{\sigma}}_{\text{Stress tensor}} \right] = \underbrace{\mathbf{f}}_{\text{External forces}}$$

Conservation of mass:

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \boldsymbol{u}$$

Rheology:

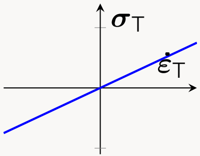
$$F(\boldsymbol{\sigma}, \underbrace{\boldsymbol{\varepsilon}}_{\text{Strain}}, \underbrace{\dot{\boldsymbol{\varepsilon}}}_{\text{Strain rate}}) = 0$$

$$\frac{\mathrm{d}\boldsymbol{\varepsilon}}{\mathrm{d}t} := \dot{\boldsymbol{\varepsilon}} := \mathrm{D}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top\right)$$

Continuum fluid mechanics

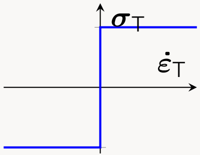
Newtonian fluids (e.g. water)

- Possibly very viscous (honey, tar)
- Always come-back to flat rest state
- Stress colinear to strain rate $\boldsymbol{\sigma} = \eta\dot{\boldsymbol{\varepsilon}}$



Yield-stress fluids (e.g. mayonnaise)

- Possibly non-zero stress with zero strain-rate
- May maintain non-flat shape



Granular continuum

Dense granular materials are yield-stress fluids

- Pressure-dependent yield-stress (Coulomb-like)
  $|\boldsymbol{\sigma}_{\mathrm{T}}| \leq \mu p$
  - Friction coefficient linked to rest angle of granular heap
- $\mu(I)$: Friction coefficient varies with “inertial number”
  - Account for relative grain size in dynamics

Continuum simulation of granular materials

As visco-plastic flows

- Most assume dense flow (do not allow grains to separate)
- "Standard" numerical methods for incompressible flows: Augmented Lagrangian or regularization
  - e.g. [Lagrée et al. 2011], [Ionescu et al. 2015]
- Computer Graphics: [Zhu and Bridson 2005]
  - [Narain et al. 2010] relaxes incompressibility

As elasto-plastic solids

- From soil mechanics
- Stress direction from elasticity
- Stiff grains: very small elasticity time-scale
- e.g. [Dunatunga et al. 2015], Computer Graphics: [Klar 2016]

Our approach

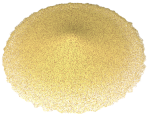
Key features

We build upon [Narain et al. 2010]:

- Inelastic approach: we assume an infinite compression Young modulus for the compacted material
- Instantaneous and implicit switching between flow regimes using hard constraints.

Our approach

Using [Narain 2010]



Our approach

Key features

We build upon [Narain et al. 2010]:

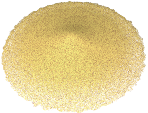
- Inelastic approach: we assume an infinite compression Young modulus for the compacted material
- Instantaneous and implicit switching between flow regimes using hard constraints.

Main differences:

- Exact Drucker–Prager frictional law

Our approach

Using [Narain 2010]



Our approach

Key features

We build upon [Narain et al. 2010]:

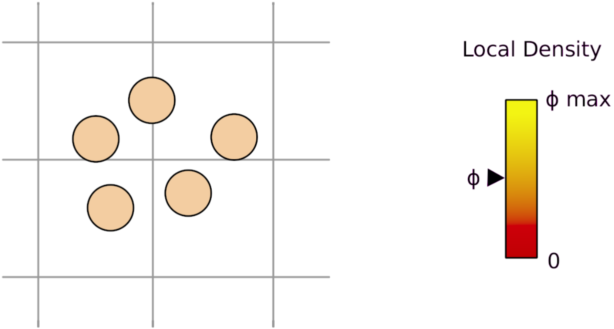
- Inelastic approach: we assume an infinite compression Young modulus for the compacted material
- Instantaneous and implicit switching between flow regimes using hard constraints.

Main differences:

- Exact Drucker–Prager frictional law
- Spatial discretization from variational formulation

Distinct regimes

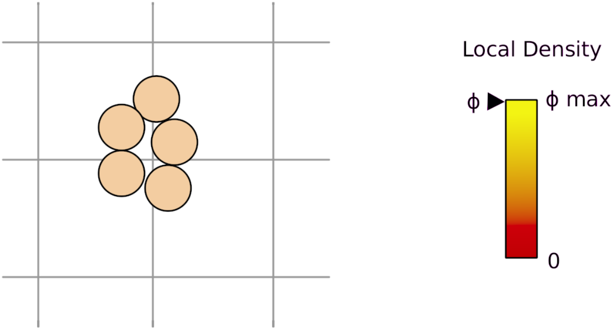
$\phi(\mathbf{x}, t)$ volume fraction field: fraction of space occupied by the grains.



$\phi < \phi_{max}$: Gaseous regime, energy dissipation through random collisions

Distinct regimes

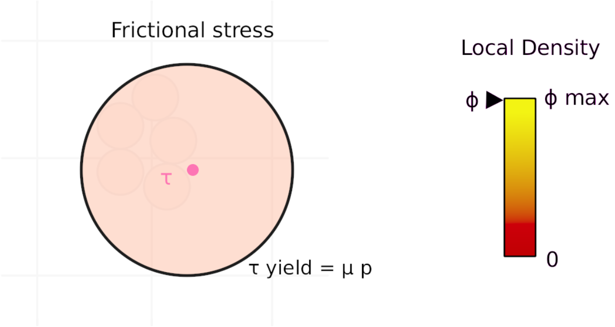
$\phi(\mathbf{x}, t)$ volume fraction field: fraction of space occupied by the grains.



$\phi = \phi_{max}$: Frictional regime, pressure-dependent yield stress.

Distinct regimes

$\phi(\mathbf{x}, t)$ volume fraction field: fraction of space occupied by the grains.

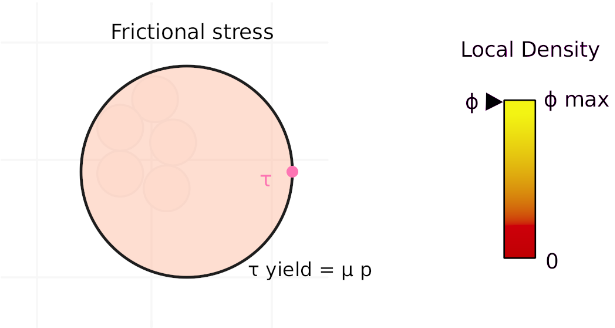


$\phi = \phi_{max}$: Frictional regime, pressure-dependent yield stress.

- Below the yield stress: solid regime

Distinct regimes

$\phi(\mathbf{x}, t)$ volume fraction field: fraction of space occupied by the grains.



$\phi = \phi_{max}$: Frictional regime, pressure-dependent yield stress.

- Below the yield stress: solid regime
- At the yield stress: liquid regime

Drucker–Prager viscoplastic rheology

$$\boldsymbol{\sigma}_{tot} = \underbrace{2\eta\dot{\boldsymbol{\varepsilon}}}_{\text{Newtonian part}} + \underbrace{\boldsymbol{\tau} - p\mathbb{I},}_{\text{Contact stress}}$$

Drucker–Prager viscoplastic rheology

$$\boldsymbol{\sigma}_{tot} = \underbrace{2\eta\dot{\boldsymbol{\varepsilon}}}_{\text{Newtonian part}} + \underbrace{\boldsymbol{\tau} - p\mathbb{I},}_{\text{Contact stress}}$$

Frictional stress $\boldsymbol{\tau}$

Drucker-Prager yield criterion with friction coefficient μ

$$\begin{cases} \boldsymbol{\tau} = \mu p \dfrac{\mathrm{Dev}\,\dot{\boldsymbol{\varepsilon}}}{|\,\mathrm{Dev}\,\dot{\boldsymbol{\varepsilon}}|} & \text{if } \mathrm{Dev}\,\dot{\boldsymbol{\varepsilon}} \neq 0 \quad \text{(Liquid)} \\ |\boldsymbol{\tau}| \leq \mu p & \text{if } \mathrm{Dev}\,\dot{\boldsymbol{\varepsilon}} = 0 \quad \text{(Rigid)} \end{cases}$$

Drucker–Prager viscoplastic rheology

$$\boldsymbol{\sigma}_{tot} = \underbrace{2\eta\dot{\boldsymbol{\varepsilon}}}_{\text{Newtonian part}} + \underbrace{\boldsymbol{\tau} - p\mathbb{I},}_{\text{Contact stress}}$$

Frictional stress $\boldsymbol{\tau}$

Drucker-Prager yield criterion with friction coefficient μ

$$\begin{cases} \boldsymbol{\tau} = \mu p \dfrac{\mathrm{Dev}\,\dot{\boldsymbol{\varepsilon}}}{|\mathrm{Dev}\,\dot{\boldsymbol{\varepsilon}}|} & \text{if } \mathrm{Dev}\,\dot{\boldsymbol{\varepsilon}} \neq 0 \quad \text{(Liquid)} \\ |\boldsymbol{\tau}| \leq \mu p & \text{if } \mathrm{Dev}\,\dot{\boldsymbol{\varepsilon}} = 0 \quad \text{(Rigid)} \end{cases}$$

Pressure p

$$0 \leq \phi_{max} - \phi \perp p \geq 0$$

(Narain et al. 2010)

Conservation equations

Conservation of mass

$$\frac{D\phi}{Dt} + \phi\nabla \cdot [\mathbf{u}] = 0$$

Conservation of momentum

$$\rho\phi\frac{D\mathbf{u}}{Dt} - \nabla \cdot \left[\phi \underbrace{\left(\eta\dot{\boldsymbol{\varepsilon}} + \boldsymbol{\tau} - p\mathbb{I}\right)}_{\boldsymbol{\sigma}_{tot}}\right] = \rho\phi\mathbf{g}$$

Time discretization

Semi-implicit integration

For each timestep Δ_t

(i) Solve momentum balance using the current volume fraction field $\phi(t)$ so that the rheology constraints hold at the end of the timestep

- Get $\mathbf{u}(t+\Delta_t)$, $p(t+\Delta_t)$, $\boldsymbol{\tau}(t+\Delta_t)$

Time discretization

Semi-implicit integration

For each timestep Δ_t

(i) Solve momentum balance using the current volume fraction field $\phi(t)$ so that the rheology constraints hold at the end of the timestep

- Get $\mathbf{u}(t + \Delta_t)$, $p(t + \Delta_t)$, $\boldsymbol{\tau}(t + \Delta_t)$

(ii) Solve the mass conservation equation using the newly computed velocity field $\mathbf{u}(t + \Delta_t)$ to get $\phi(t + \Delta_t)$

- Hybrid method: move particles
- We use APIC: Affine Particle-in-Cell [Jiang et al. 2015]

Discrete-time rheology

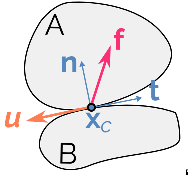
Linearizing the change in volume fraction over the timestep

- $\boldsymbol{\lambda} := p\mathbb{I} - \boldsymbol{\tau}$ homogeneous to a stress
- $\boldsymbol{\gamma} := \phi(t)\dot{\boldsymbol{\varepsilon}} + \frac{1}{d}\frac{\phi_{\max} - \phi(t)}{\Delta_t}\mathbb{I}$ homogeneous to a strain rate

Discrete-time rheology

Linearizing the change in volume fraction over the timestep

- $\boldsymbol{\lambda} := p\mathbb{I} - \boldsymbol{\tau}$ homogeneous to a stress
- $\boldsymbol{\gamma} := \phi(t)\dot{\boldsymbol{\varepsilon}} + \frac{1}{d}\frac{\phi_{\max} - \phi(t)}{\Delta_t}\mathbb{I}$ homogeneous to a strain rate



Gaseous

$$\begin{cases} \boldsymbol{\gamma} \geq 0 \\ \boldsymbol{\lambda} = 0 \end{cases}$$

Solid

$$\begin{cases} \boldsymbol{\gamma} = 0 \\ \boldsymbol{\lambda} \in K_\mu \end{cases}$$

Liquid

$$\begin{cases} \operatorname{Tr} \boldsymbol{\gamma} = 0 \\ \boldsymbol{\lambda} \in \partial K_\mu \\ \operatorname{Dev} \boldsymbol{\lambda} = -\alpha \operatorname{Dev} \boldsymbol{\gamma}, \ \alpha \geq 0 \end{cases}$$

Equivalent to Signorini-Coulomb frictional contact law in discrete mechanics.
($\boldsymbol{\lambda} \sim \mathbf{f}$ force, $\boldsymbol{\gamma} \sim \boldsymbol{u}$ relative velocity, Tr $\sim$ normal part, Dev $\sim$ tangential part)

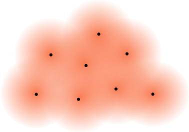
Spatial discretization

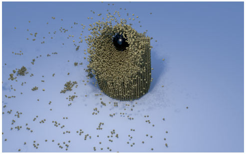
We must restrict ourselves to a limited number of degrees of freedom for:

- Scalar volume fraction field
- Vector velocity field
- Symmetric tensor stress and strain field

Spatial discretization

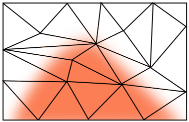
Particle-based methods (e.g. SPH)

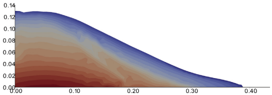




[Alduán & Otaduy 2011]

Mesh-based methods (e.g. FEM)

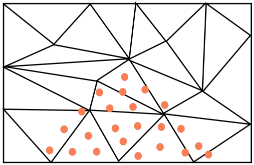


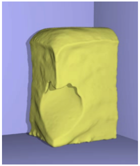


[Ionescu et al. 2015]

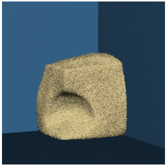
Spatial discretization

Hybrid methods: Particles for material state + mesh for velocities





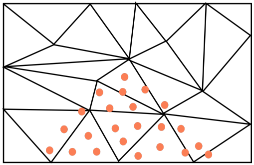
[Zhu and Bridson 2005]

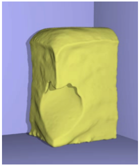


[Narain et al. 2010]

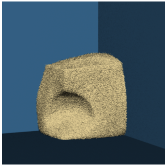
Spatial discretization

Hybrid methods: Particles for material state + mesh for velocities

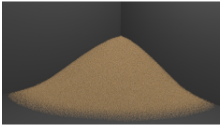




[Zhu and Bridson 2005]



[Narain et al. 2010]



[Klar et al. 2016]

We use the Material Point Method (MPM)

- For granulars: [Wieckowski 1999], [Dunatunga et al. 2015], [Klar et al. 2016] (concurrently to this work)

MPM: Principle

Volume fraction field ϕ discretized as a sum of Dirac point masses:

$$\phi(\mathbf{x}) = \sum_p \underbrace{V_p}_{\text{Particle volume}} \delta(\mathbf{x} - \underbrace{\mathbf{x}_p}_{\text{Particle position}})$$

Integration over the simulation domain Ω:

$$\int_\Omega \phi \mathbf{v} = \sum_p V_p \mathbf{v}(\mathbf{x}_p)$$

MPM: Principle

Volume fraction field ϕ discretized as a sum of Dirac point masses:

$$\phi(\mathbf{x}) = \sum_p \underbrace{V_p}_{\text{Particle volume}} \delta(\mathbf{x} - \underbrace{\mathbf{x}_p}_{\text{Particle position}})$$

Integration over the simulation domain Ω:

$$\int_\Omega \phi \mathbf{v} = \sum_p V_p \mathbf{v}(\mathbf{x}_p)$$

Interpretation: $\mathbf{x}_p \sim$ quadrature points and V_p corresponding weights

MPM: Application

Weak momentum balance

$$\frac{\rho}{\Delta_t}\phi\mathbf{u} - \nabla\cdot\left[\phi\left(\eta\mathrm{D}\left(\mathbf{u}\right)-\boldsymbol{\lambda}\right)\right] = \rho\phi\mathbf{f}$$

MPM: Application

Weak momentum balance

$$\frac{\rho}{\Delta_t}\phi\mathbf{u} - \nabla\cdot\left[\phi\left(\eta \mathrm{D}\left(\mathbf{u}\right) - \boldsymbol{\lambda}\right)\right] = \rho\phi\mathbf{f}$$

FEM: multiplying by a test function **v** and integrating over Ω + Green formula:

$$\int_\Omega \frac{\rho}{\Delta_t}\phi\mathbf{u}.\mathbf{v} + \int_\Omega \phi\left(\eta\mathrm{D}\left(\mathbf{u}\right) - \boldsymbol{\lambda}\right) : \mathrm{D}\left(\mathbf{v}\right) = \int_\Omega \rho\phi\mathbf{f}.\mathbf{v} \qquad \forall\mathbf{v}$$

MPM: Application

Weak momentum balance

$$\frac{\rho}{\Delta_t}\phi\mathbf{u} - \nabla\cdot\left[\phi\left(\eta\mathrm{D}\left(\mathbf{u}\right)-\boldsymbol{\lambda}\right)\right] = \rho\phi\mathbf{f}$$

FEM: multiplying by a test function **v** and integrating over Ω + Green formula:

$$\int_\Omega \frac{\rho}{\Delta_t}\phi\mathbf{u}.\mathbf{v} + \int_\Omega \phi\left(\eta\mathrm{D}\left(\mathbf{u}\right)-\boldsymbol{\lambda}\right):\mathrm{D}\left(\mathbf{v}\right) = \int_\Omega \rho\phi\mathbf{f}.\mathbf{v} \qquad \forall\mathbf{v}$$

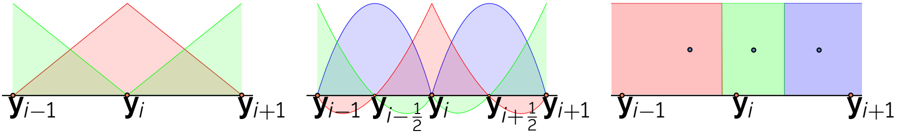
MPM: $\phi(\mathbf{x}) = \sum_p V_p\delta(\mathbf{x}-\mathbf{x}_p)$

$$\sum_p V_p\left(\tfrac{\rho}{\Delta_t}\mathbf{u}.\mathbf{v} + \eta\mathrm{D}\left(\mathbf{u}\right):\mathrm{D}\left(\mathbf{v}\right)\right)(\mathbf{x}_p) - \sum_p V_p(\boldsymbol{\lambda}:\mathrm{D}\left(\mathbf{v}\right))(\mathbf{x}_p) = \rho\sum_p V_p(\mathbf{f}.\mathbf{v})(\mathbf{x}_p) \quad \forall\mathbf{v}$$

Basis functions

We still need to discretize $\mathbf{u}$ (velocity, vector) and $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ (stress / strain, tensors) using a finite number of degrees of freedom (grid nodes).

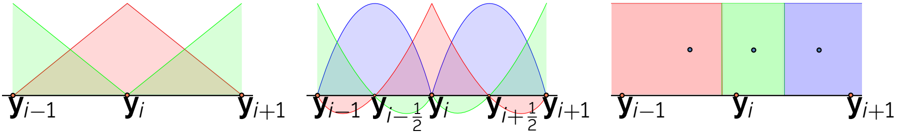
$$\mathbf{u}(\mathbf{x}) = \sum_i N_i^{\mathbf{v}}(\mathbf{x})\underline{\mathbf{u}}_i, \qquad \underline{\mathbf{u}}_i = \mathbf{u}(\mathbf{y}_i), \qquad (\mathbf{y}_i)\text{degrees of freedom}$$



Basis functions

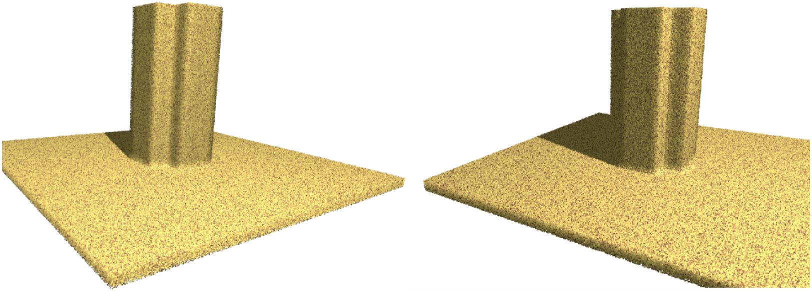
We still need to discretize $\mathbf{u}$ (velocity, vector) and $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ (stress / strain, tensors) using a finite number of degrees of freedom (grid nodes).

$$\mathbf{u}(\mathbf{x}) = \sum_i N_i^{\mathbf{v}}(\mathbf{x})\underline{\mathbf{u}}_i, \qquad \underline{\mathbf{u}}_i = \mathbf{u}(\mathbf{y}_i), \qquad (\mathbf{y}_i)\text{degrees of freedom}$$



- Affects well-posedness of the numerical system, spatial convergence and computational performance
- May create visual artifacts

Cohesion



Trilinear stresses & cohesion decay

Particle-based stresses, no decay

Discrete System

Concatenating all unknown components and writing constraints at stress quadrature points leads to

$$\begin{cases} A\underline{\mathbf{u}} = \underline{\mathbf{f}} + B^\top \underline{\boldsymbol{\lambda}} & \text{(Momentum balance)} \\ \underline{\gamma} = B\underline{\mathbf{u}} + \underline{\mathbf{k}} & \text{(Strain from velocity)} \\ (\underline{\gamma}_i, \underline{\boldsymbol{\lambda}}_i) \in \mathcal{DP}(\mu) \quad \forall i = 1 \dots n & \text{(Strain–stress relationship)} \end{cases}$$

- Similar to discrete contact mechanics with Coulomb friction
  - …in dimension 6
  - …A^{-1} may be dense
    - use of proximal or interior-point algorithms
    - or low-Newtonian viscosity approximation

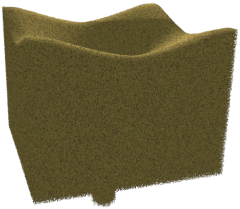
Discrete System

Concatenating all unknown components and writing constraints at stress quadrature points leads to

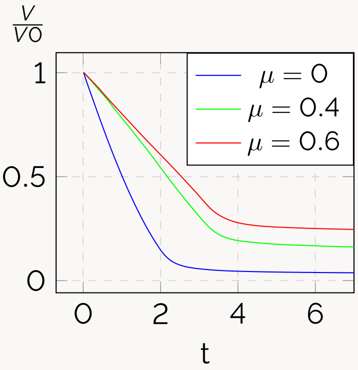
$$\begin{cases} A\underline{\mathbf{u}} = \underline{\mathbf{f}} + B^\top \underline{\boldsymbol{\lambda}} & \text{(Momentum balance)} \\ \underline{\gamma} = B\underline{\mathbf{u}} + \underline{\mathbf{k}} & \text{(Strain from velocity)} \\ (\underline{\gamma}_i, \underline{\boldsymbol{\lambda}}_i) \in \mathcal{DP}(\mu) \quad \forall i = 1 \dots n & \text{(Strain–stress relationship)} \end{cases}$$

- Similar to discrete contact mechanics with Coulomb friction
  - …in dimension 6
  - …A^{-1} may be dense
    - use of proximal or interior-point algorithms
    - or low-Newtonian viscosity approximation
  - In practice: Matrix-free Gauss–Seidel solver with Fischer-Burmeister local solver

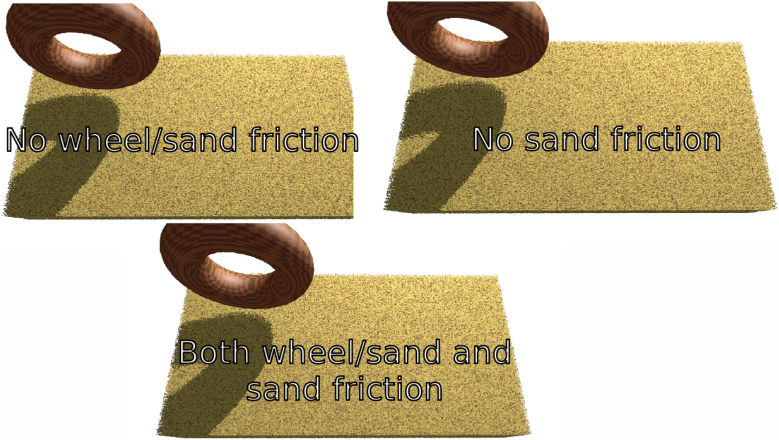
Silo



Constant discharge rate



Rigid-body coupling



Conclusion

- Very stable simulations at a reasonable computational cost

Perspectives

- Explore other shape functions to improve
  - Volume preservation
  - Visual artifacts in degenerate cases
- Interactions with surrounding fluid (air, water)

Outline

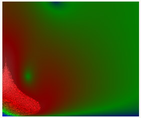
1. Efficient simulation of frictional contacts in DEM

- Application to hair simulation
- Presented at Siggraph Asia 2011

2. Continuum simulation of dry granular materials

- Dense case: JNNFM 2016
- General case: Siggraph 2016

3. Continuum simulation of granular materials in a Newtonian fluid



- Exploratory 2D work
- Submitted to "Powder and Grains 2017"

Diphasic simulation

Motivation

- Qualitative effects of Newtonian fluid on granular collapse
- Assume Drucker–Prager still holds at maximal volume fraction
- Phase velocities must differ to allow compression

Two-velocities model

- Conservation of momentum and mass for each phase
- Interactions terms:
  - Stokes drag: $\mathbf{f}^d = \eta(\phi)(\mathbf{u}_f - \mathbf{u}_g)$
  - Buoyancy

Diphasic simulation

Motivation

- Qualitative effects of Newtonian fluid on granular collapse
- Assume Drucker–Prager still holds at maximal volume fraction
- Phase velocities must differ to allow compression

Two-velocities model

- Conservation of momentum and mass for each phase
- Interactions terms:
  - Stokes drag: $\mathbf{f}^d = \eta(\phi)(\mathbf{u}_f - \mathbf{u}_g)$
  - Buoyancy
- Discrete problem: DCFP with linear constraints

Results

Concluding Remarks

Contributions

Efficient treatment of friction contact in hair dynamics

- New one-contact solvers
- Miscellaneous refinements of Gauss–Seidel and Projected-Gradient methods

Non-smooth simulation of dry granular flows

- Leveraging tools from discrete contact dynamics
- Taking into account different regimes

Non-smooth simulation of diphasic granular flows

- Model and numerical method

Conclusion

- Dry friction necessary for realism
- Implicit handling of rigid frictional contacts
  - No jittering or creeping motion
  - Better numerical conditionning
  - Avoids having to simulate elasticity timescale
- Non-smooth contact dynamics directly applicable to continuum simulation
  - Similar modeling framework
  - Same discrete problem structure

Conclusion

- Dry friction necessary for realism
- Implicit handling of rigid frictional contacts
  - No jittering or creeping motion
  - Better numerical conditionning
  - Avoids having to simulate elasticity timescale
- Non-smooth contact dynamics directly applicable to continuum simulation
  - Similar modeling framework
  - Same discrete problem structure

Perspectives

- Continuum model for hair dynamics
- Scalable DCFP solver

Publications

Peer-reviewed journals

- “A hybrid iterative solver for robustly capturing Coulomb friction in hair dynamics”, Siggraph Asia 2011 **G. Daviet**, F. Bertails-Descoubes, L. Boissieux
- “Inverse dynamic hair modeling with frictional contact”, Siggraph Asia 2013 A. Derouet-Jourdan, F. Bertails-Descoubes, **G. Daviet**, J. Thollot
- “Nonsmooth simulation of dense granular flows with pressure-dependent yield stress”, Journal of Non-Newtonian Fluid Mechanics **G. Daviet**, F. Bertails-Descoubes
- “A Semi-Implicit Material Point Method for the Continuum Simulation of Granular Materials”, Siggraph 2016 **G. Daviet**, F. Bertails-Descoubes

Posters & non-reviewed reports

- “Quartic formulation of Coulomb 3D frictional contact”, Inria Tech Report, 2011, O. Bonnefon, **G. Daviet**
- “Fast cloth simulation with implicit contact and exact coulomb friction”, SCA 2015 Poster, **G. Daviet**, F. Bertails-Descoubes, R. Casati
- “Inverse Elastic Cloth Design with Contact and Friction”, Inria Tech Report, 2015 R.Casati, **G. Daviet**, F. Bertails-Descoubes
- “Simulation of Drucker–Prager granular flows inside Newtonian fluids” (submitted) **G. Daviet**, F. Bertails-Descoubes

Thank you for your attention

Many thanks for the insightful discussions:

- Laurence Boissieux
- Romain Casati
- Emmanuel Delangre
- Alexandre Derouet-Jourdan
- Pierre-Yves Lagrée
- Pierre Saramito

PERSYVAL-Lab

This work has been partially supported by the LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01) funded by the French program Investissement d'avenir