Supplemental Material for:
A Semi-Implicit Material Point Method for the Continuum Simulation of Granular Materials

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1 Energy considerations

We can decompose the total energy of the system as the sum of its potential and kinetic energies, i.e., $E_t = E_p + E_c$, with

$$E_p := -\int_\Omega \rho\phi\,(\mathbf{x}\cdot\boldsymbol{g})$$
$$E_c := \int_\Omega \frac{1}{2}\rho\phi\,(\mathbf{u}\cdot\mathbf{u})\,.$$

Since we are using homogeneous Dirichlet boundary conditions, their evolution follows

$$\frac{dE_p}{dt} = -\rho\int_\Omega \frac{\partial\phi}{\partial t}(\mathbf{x}\cdot\boldsymbol{g}) = \rho\int_\Omega \nabla\cdot[\phi\mathbf{u}]\,(\mathbf{x}\cdot\boldsymbol{g})$$
$$= -\rho\int_\Omega \phi\,(\mathbf{u}\cdot\mathbf{g}) + \rho\underbrace{\int_{\partial\Omega}\phi\,(\mathbf{u}\cdot\boldsymbol{n}_\Omega)\,(\boldsymbol{x}\cdot\mathbf{g})}_{=0}$$

and

$$\frac{dE_c}{dt} = \int_\Omega \rho\phi\left(\frac{D\mathbf{u}}{Dt}\cdot\mathbf{u}\right)$$
$$= \int_\Omega \eta\,(\nabla\cdot[\phi\dot{\boldsymbol{\varepsilon}}]\cdot\mathbf{u}) - (\nabla\cdot[\phi\boldsymbol{\lambda}]\cdot\mathbf{u}) + \rho\phi\,(\boldsymbol{g}\cdot\mathbf{u})$$
$$= \rho\int_\Omega \phi\,(\mathbf{u}\cdot\mathbf{g}) - \eta\underbrace{\int_\Omega \phi\dot{\boldsymbol{\varepsilon}}:\dot{\boldsymbol{\varepsilon}}}_{\geq 0} + \int_\Omega \phi\boldsymbol{\lambda}:\dot{\boldsymbol{\varepsilon}}$$
$$+ \underbrace{\int_{\partial\Omega}\phi\,(\mathbf{u}\cdot(\eta\dot{\boldsymbol{\varepsilon}}-\boldsymbol{\lambda})\,\boldsymbol{n}_\Omega)}_{=0}\,.$$

Therefore $\frac{dE_t}{dt} \leq \int_\Omega \phi\boldsymbol{\lambda}:\dot{\boldsymbol{\varepsilon}}$. For the system to be dissipative, it suffices that $(\boldsymbol{\lambda};\boldsymbol{\gamma}) \in \mathcal{DP}(\hat{\mu}) \implies \phi\boldsymbol{\lambda}:\dot{\boldsymbol{\varepsilon}} \leq 0$, which can be easily verified.

Indeed, let $(\boldsymbol{\lambda};\boldsymbol{\gamma}) \in \mathcal{DP}(\hat{\mu})$, we will show that

$$\begin{cases} \operatorname{Dev}\boldsymbol{\lambda} : \operatorname{Dev}(\phi\dot{\boldsymbol{\varepsilon}}) \leq 0 & \text{(a)} \\ \operatorname{Tr}\boldsymbol{\lambda}\operatorname{Tr}(\phi\dot{\boldsymbol{\varepsilon}}) \leq 0 & \text{(b)} \end{cases}$$

Since $\operatorname{Dev}(\phi\dot{\boldsymbol{\varepsilon}}) = \operatorname{Dev}\boldsymbol{\gamma}$, (a) is trivial; either $\operatorname{Dev}(\phi\dot{\boldsymbol{\varepsilon}}) = 0$, or $\operatorname{Dev}(\phi\dot{\boldsymbol{\varepsilon}}) \neq 0$ and $\operatorname{Dev}\boldsymbol{\lambda} = -\left(\mu\frac{\operatorname{Tr}\boldsymbol{\lambda}}{\sqrt{6}}\right)\frac{\operatorname{Dev}\boldsymbol{\gamma}}{|\operatorname{Dev}\boldsymbol{\gamma}|}$ with $\operatorname{Tr}\boldsymbol{\lambda} \geq 0$.

Now, for (a), the complementarity condition implies either $\operatorname{Tr}\boldsymbol{\lambda} = 0$, or‘ $\operatorname{Tr}\boldsymbol{\lambda} > 0$ and $\operatorname{Tr}\boldsymbol{\gamma} = 0$. Since $\beta \geq 0$, $\operatorname{Tr}(\phi\dot{\boldsymbol{\varepsilon}}) \leq 0$, and $\operatorname{Tr}\boldsymbol{\lambda}\operatorname{Tr}(\phi\dot{\boldsymbol{\varepsilon}}) \leq 0$.

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2 Derivation of variational formulation

Multiplying both sides of the definition of $\boldsymbol{\gamma}$ from Equation (8) by a test function $\boldsymbol{\tau}$ and integrating over Ω yields

$$\int_\Omega \boldsymbol{\gamma}:\boldsymbol{\tau} = \int_\Omega \phi\dot{\boldsymbol{\varepsilon}}:\boldsymbol{\tau} + \int_\Omega \frac{\phi_{max}-\phi}{3\Delta_t}\mathbf{I}:\boldsymbol{\tau}$$
$$= \sum V_p\left(\boldsymbol{\tau}(\boldsymbol{x}_p^n):\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}_p^n) - \frac{\mathbf{I}:\boldsymbol{\tau}(\boldsymbol{x}_p^n)}{3\Delta_t}\right) + \int_\Omega \frac{\phi_{max}}{3\Delta_t}\mathbf{I}:\boldsymbol{\tau}$$
$$= b(\boldsymbol{\tau},\mathbf{u}) + k(\boldsymbol{\tau}).$$

We retrieve Equation (11), $s(\boldsymbol{\gamma},\boldsymbol{\tau}) = b(\boldsymbol{\tau},\mathbf{u}) + k(\boldsymbol{\tau})$.

We can proceed similarly for the discrete-time momentum balance equation (9),

$$\frac{\rho}{\Delta_t}\phi\mathbf{u} + \nabla\cdot[\phi\,(\boldsymbol{\lambda}-\eta\dot{\boldsymbol{\varepsilon}})] = \rho\phi\left(\boldsymbol{g} + \frac{\mathbf{u}^{p\to g}}{\Delta_t}\right),$$

and get

$$\int_\Omega\left(\frac{\rho}{\Delta_t}\phi\mathbf{u}\cdot\mathbf{v}\right) = m(\mathbf{u},\mathbf{v})$$
$$\int_\Omega\left(\rho\phi\left(\boldsymbol{g}+\frac{\mathbf{u}^{p\to g}}{\Delta_t}\right)\cdot\mathbf{v}\right) = l(\mathbf{v}).$$

Using the Stokes formula with our homogeneous Dirichlet boundary conditions, we also have

$$\int_\Omega(\nabla\cdot[\phi\boldsymbol{\lambda}-\eta\phi\dot{\boldsymbol{\varepsilon}}]\cdot\mathbf{v}) = \int_\Omega(\eta\phi\dot{\boldsymbol{\varepsilon}}-\phi\boldsymbol{\lambda}):\mathrm{D}(\mathbf{v})$$
$$= \sum V_p\left(\eta\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}_p^n) - \boldsymbol{\lambda}(\boldsymbol{x}_p^n)\right):\mathrm{D}(\mathbf{v})(\boldsymbol{x}_p^n)$$
$$= a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},\boldsymbol{\lambda}).$$

and retrieve Equation (10), $m(\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) = b(\boldsymbol{\lambda},\mathbf{v}) + l(\mathbf{v})$.

3 Frictional boundary condition

Suppose that $(\boldsymbol{\lambda}_{\mathrm{RB}};\boldsymbol{\gamma}_{\mathrm{RB}}) \in \mathcal{DP}(\mu_{\mathrm{RB}})$, with $\boldsymbol{\gamma}_{\mathrm{RB}} := \frac{1}{2}\left(\overline{\boldsymbol{v}}\boldsymbol{n}_{\mathrm{RB}}^\top + \boldsymbol{n}_{\mathrm{RB}}\overline{\boldsymbol{v}}^\top\right)$, and $\|\boldsymbol{n}_{\mathrm{RB}}\| = 1$.

The force induced by a stress $\boldsymbol{\sigma}$ through a plane with normal $\boldsymbol{n}$ is computed as $\boldsymbol{\sigma}\boldsymbol{n}$; the reaction force induced by the material on the frictional boundary is therefore $\boldsymbol{r} = \boldsymbol{\lambda}_{\mathrm{RB}}\boldsymbol{n}_{\mathrm{RB}}$. In the following, we investigate the relationship between $\boldsymbol{r}$ and $\overline{\boldsymbol{v}}$, the relative velocity of the boundary w.r.t. the granular material.

3.1 Signorini condition

First, remark that $\operatorname{Tr}\boldsymbol{\gamma}_{\mathrm{RB}} = (\overline{\boldsymbol{v}}\cdot\boldsymbol{n}_{\mathrm{RB}}) = \overline{\boldsymbol{v}}_{\mathrm{N}}$,

$$\boldsymbol{r}_{\mathrm{N}} = \frac{1}{3}(\operatorname{Tr}\boldsymbol{\lambda}_{\mathrm{RB}}) + (\operatorname{Dev}\boldsymbol{\lambda}_{\mathrm{RB}}):(\boldsymbol{n}_{\mathrm{RB}}\boldsymbol{n}_{\mathrm{RB}}^\top)$$

and

$$\begin{aligned}(\operatorname{Dev} \boldsymbol{\lambda}_{\text{RB}}) : (\boldsymbol{n}_{\text{RB}}\boldsymbol{n}_{\text{RB}}^\top) &\leq 2\,|\operatorname{Dev} \boldsymbol{\lambda}_{\text{RB}}||\boldsymbol{n}_{\text{RB}}\boldsymbol{n}_{\text{RB}}^\top| \\ &\leq \frac{\mu_{\text{RB}}}{\sqrt{6}}\operatorname{Tr}\boldsymbol{\lambda}_{\text{RB}} \times \frac{2}{\sqrt{2}} \\ &\leq \frac{\mu_{\text{RB}}}{\sqrt{3}}\operatorname{Tr}\boldsymbol{\lambda}_{\text{RB}}\,.\end{aligned}$$

Therefore,

$$\left(\frac{1-\sqrt{3}\mu_{\text{RB}}}{3}\right)\operatorname{Tr}\boldsymbol{\lambda}_{\text{RB}} \leq \boldsymbol{r}_{\text{N}} \leq \left(\frac{1+\sqrt{3}\mu_{\text{RB}}}{3}\right)\operatorname{Tr}\boldsymbol{\lambda}_{\text{RB}}\,.$$

For $\mu_{\text{RB}} < \frac{1}{\sqrt{3}}$, we thus have

$$0 \leq \operatorname{Tr}\boldsymbol{\gamma}_{\text{RB}} \perp \operatorname{Tr}\boldsymbol{\lambda}_{\text{RB}} \geq 0 \implies 0 \leq \overline{\boldsymbol{v}}_{\text{N}} \perp \boldsymbol{r}_{\text{N}} \geq 0\,,$$

i.e., the Signorini condition is satisfied.

3.2 Tangential reaction

Notice that

$$\begin{aligned}\operatorname{Dev}(\boldsymbol{\gamma}_{\text{RB}})\boldsymbol{n}_{\text{RB}} &= \frac{1}{2}\overline{\boldsymbol{v}} + (\frac{1}{2}-\frac{1}{3})\,(\overline{\boldsymbol{v}}\cdot\boldsymbol{n}_{\text{RB}})\,\boldsymbol{n}_{\text{RB}} \\ &= \frac{1}{2}\overline{\boldsymbol{v}}_{\text{T}} + \frac{2}{3}\overline{\boldsymbol{v}}_{\text{N}}\boldsymbol{n}\,.\end{aligned}$$

Sliding case First suppose that $\overline{\boldsymbol{v}}_{\text{T}} \neq 0$, therefore $\operatorname{Dev}\boldsymbol{\gamma}_{\text{RB}} \neq 0$, and $\mathcal{DP}\,(\mu_{\text{RB}})$ imposes that $\operatorname{Dev}\boldsymbol{\lambda}_{\text{RB}} = -\alpha\operatorname{Dev}\boldsymbol{\gamma}_{\text{RB}}$, $\alpha > 0$. Since $\boldsymbol{r} = \frac{1}{3}\operatorname{Tr}\boldsymbol{\lambda}_{\text{RB}}\boldsymbol{n} + \operatorname{Dev}\boldsymbol{\lambda}_{\text{RB}}\boldsymbol{n}$, we can identity that

$$\boldsymbol{r}_{\text{T}} = -\frac{1}{2}\alpha\overline{\boldsymbol{v}}_{\text{T}}$$

i.e. the tangential friction force is opposed to the tangential relative velocity. Now, let us show that $\boldsymbol{r}$ lies on the boundary of the second-order cone of aperture $\sqrt{\frac{3}{2}}\mu_{\text{RB}}$, i.e., $\|\boldsymbol{r}_{\text{T}}\| = \sqrt{\frac{3}{2}}\mu_{\text{RB}}\boldsymbol{r}_{\text{N}}$.

From the Signori condition, $\overline{\boldsymbol{v}}_{\text{N}} > 0$ implies $\boldsymbol{r}_{\text{N}} = 0$, and therefore $\operatorname{Tr}\boldsymbol{\lambda}_{\text{RB}} = 0$. This means $|\operatorname{Dev}\boldsymbol{\lambda}_{\text{RB}}| = 0$, and consequently $\|\boldsymbol{r}_{\text{T}}\|$. Our relation $\|\boldsymbol{r}_{\text{T}}\| = \sqrt{\frac{3}{2}}\mu_{\text{RB}}\boldsymbol{r}_{\text{N}}$ is trivially satisfied.

We now have to study the case $\overline{\boldsymbol{v}}_{\text{N}} = 0$. $\operatorname{Dev}\boldsymbol{\lambda}_{\text{RB}} = -\alpha\operatorname{Dev}\boldsymbol{\gamma}_{\text{RB}}$ means that

$$\|\boldsymbol{r}_{\text{T}}\| = \|\operatorname{Dev}\boldsymbol{\lambda}_{\text{RB}}\boldsymbol{n}\| = |\operatorname{Dev}\boldsymbol{\lambda}_{\text{RB}}\boldsymbol{n}|\frac{\|\operatorname{Dev}\boldsymbol{\gamma}_{\text{RB}}\boldsymbol{n}_{\text{RB}}\|}{|\operatorname{Dev}\boldsymbol{\gamma}_{\text{RB}}|}\,.$$

Since $\overline{\boldsymbol{v}}.\boldsymbol{n}_{\text{RB}} = 0$,

$$\|\operatorname{Dev}(\boldsymbol{\gamma}_{\text{RB}})\boldsymbol{n}_{\text{RB}}\| = \frac{1}{2}\|\overline{\boldsymbol{v}}_{\text{T}}\|$$

and

$$\begin{aligned}|\operatorname{Dev}(\boldsymbol{\gamma}_{\text{RB}})|^2 = |\boldsymbol{\gamma}_{\text{RB}}|^2 &= |\overline{\boldsymbol{v}}\boldsymbol{n}_{\text{RB}}^\top|^2 - \frac{1}{4}|\overline{\boldsymbol{v}}\boldsymbol{n}_{\text{RB}}^\top - \boldsymbol{n}_{\text{RB}}\overline{\boldsymbol{v}}^\top|^2 \\ &= \frac{1}{2}\|\overline{\boldsymbol{v}}\|^2 - \frac{1}{4}\|\overline{\boldsymbol{v}}\wedge\boldsymbol{n}_{\text{RB}}\|^2 = \frac{1}{2}\|\overline{\boldsymbol{v}}\|^2 - \frac{1}{4}\|\overline{\boldsymbol{v}}_{\text{T}}\|^2 \\ &= \frac{1}{4}\|\overline{\boldsymbol{v}}_{\text{T}}\|^2 = \|\operatorname{Dev}(\boldsymbol{\gamma}_{\text{RB}})\boldsymbol{n}_{\text{RB}}\|^2\,.\end{aligned}$$

This means

$$\|\boldsymbol{r}_{\text{T}}\| = |\operatorname{Dev}\boldsymbol{\lambda}_{\text{RB}}| = \frac{\mu_{\text{RB}}}{\sqrt{6}}\operatorname{Tr}\boldsymbol{\lambda}_{\text{RB}} = \sqrt{\frac{3}{2}}\mu_{\text{RB}}\boldsymbol{r}_{\text{N}}\,.$$

The sliding case thereforce satisfies the Coulomb law with coefficient $\sqrt{\frac{3}{2}}\mu_{\text{RB}}$.

Sticking When $\overline{\boldsymbol{v}} = 0$, we cannot conclude without more information about the relationship between $\boldsymbol{\lambda}_{\text{RB}}$ and $\boldsymbol{\gamma}_{\text{RB}}$. Indeed, we can only verify that

$$\begin{aligned}\|\boldsymbol{r}_{\text{T}}\| &\leq \|\operatorname{Dev}\boldsymbol{\lambda}_{\text{RB}}\boldsymbol{n}\| \leq \sqrt{2}|\operatorname{Dev}\boldsymbol{\lambda}_{\text{RB}}| \\ &\leq \frac{\mu_{\text{RB}}}{\sqrt{3}}\operatorname{Tr}\boldsymbol{\lambda}_{\text{RB}} \leq \frac{\sqrt{3}\mu_{\text{RB}}}{1-\sqrt{3}\mu_{\text{RB}}}\boldsymbol{r}_{\text{N}}\end{aligned}$$

i.e the reaction force has to lie inside a second-order cone of aperture $\frac{\sqrt{3}\mu_{\text{RB}}}{1-\sqrt{3}\mu_{\text{RB}}}$.

This last bound does not correspond to the one derived for the sliding case (except when $\mu_{\text{RB}} = 0$), but nevertheless models a coupling between the tangential and normal reaction forces.

3.3 Reverse inclusion

For any $(\boldsymbol{r};\overline{\boldsymbol{v}}) \in \mathcal{C}^3\left(\sqrt{\frac{3}{2}}\mu\right)$ – i.e., satisfying the 3D Coulomb law with friction coefficient $\sqrt{\frac{3}{2}}\mu$ – we can construct a symmetric tensor $\boldsymbol{\lambda}_{\text{RB}}$ such that $(\boldsymbol{\lambda}_{\text{RB}};\boldsymbol{\gamma}_{\text{RB}}) \in \mathcal{DP}\,(\mu)$. Indeed, let

$$\boldsymbol{\lambda}_{\text{RB}} := \left(\boldsymbol{r}_{\text{T}}\boldsymbol{n}_{\text{RB}}^\top + \boldsymbol{n}_{\text{RB}}\boldsymbol{r}_{\text{T}}^\top\right) + \boldsymbol{r}_{\text{N}}\mathbf{I}$$

We have

$$\begin{aligned}\operatorname{Tr}\boldsymbol{\lambda}_{\text{RB}} &= 3\boldsymbol{r}_{\text{N}} = \sqrt{6}\sqrt{\frac{3}{2}}\boldsymbol{r}_{\text{N}} \\ \boldsymbol{\lambda}_{\text{RB}}\boldsymbol{n}_{\text{RB}} &= \boldsymbol{r}_{\text{T}} + \boldsymbol{n}_{\text{RB}}(\boldsymbol{r}_{\text{N}}) = \boldsymbol{r} \\ \operatorname{Dev}(\boldsymbol{\lambda}_{\text{RB}})\boldsymbol{n}_{\text{RB}} &= \boldsymbol{r}_{\text{T}} \\ |\operatorname{Dev}(\boldsymbol{\lambda}_{\text{RB}})| &= \left|\boldsymbol{r}_{\text{T}}\boldsymbol{n}_{\text{RB}}^\top + \boldsymbol{n}_{\text{RB}}(\boldsymbol{r}-\boldsymbol{r}_{\text{N}}\boldsymbol{n}_{\text{RB}})^\top\right| \\ &= \|\boldsymbol{r}_{\text{T}}\|\end{aligned}$$

It can be easily verified that for any case of the $\mathcal{C}^3\left(\sqrt{\frac{3}{2}}\mu\right)$ disjunctive formulation satisfied by $\boldsymbol{r}$ and $\overline{\boldsymbol{v}}$, the corresponding case of $\mathcal{DP}\,(\mu)$ is satisfied by $(\boldsymbol{\lambda}_{\text{RB}};\boldsymbol{\gamma}_{\text{RB}})$.